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q-deformed SUSY algebra for a $sl_q(n|n)$-covariant q-boson and q-fermion system

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Abstract. The q-deformed $N = 2$ SUSY algebra is obtained for a $sl_q(n|n)$-covariant q-boson and q-fermion system and the Hamiltonian for q-bosons and q-fermions is constructed.

1. Introduction

Lie algebra enables us to describe physical systems. Its elegant applications were found in quantum mechanics within the concept of spectrum-generating algebras. The most famous example is given by the harmonic oscillator problem where the spectrum is generated by the Heisenberg–Weyl algebra. Some time ago much attention was drawn to the deformation of Lie algebras which are called quantum algebras or quantum groups [1, 2].

Quantum groups or q-deformed Lie algebra imply some specific deformations of classical Lie algebras. From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

Biedenharn [3] and Macfarlane [4] introduced q-deformed harmonic oscillator algebras which are sometimes called the q-boson algebra. After this was done, the q-deformation of an ordinary fermion algebra was obtained [5].

Since the $N = 2$ SUSY algebra for ordinary (undeformed) bosons and ordinary fermions was constructed by Witten [6], the q-deformed $N = 2$ SUSY algebra [7, 8] has been constructed by using the q-deformation of boson (or fermion) algebra. Most of q-deformed $N = 2$ SUSY algebra was obtained by assuming that the (q-deformed) bosons and (q-deformed) fermions are mutually independent, which implies that their step operators are mutually commuting. However, if we consider the idea of a q-superplane [9–11] with $sl_q(n|n)$ covariance, the above-mentioned assumption is improper for correct q-deformation of $N = 2$ SUSY for q-deformed bosons and q-deformed fermions. In the q-superplane, the bosonic coordinate (x) and fermionic coordinate (θ) are not mutually commuting but they obey the q-commutation relation

$$x\theta = \sqrt{q}\theta x$$

with the nilpotency of a fermionic coordinate.

The idea of a q-superplane enables us to construct the multimode q-oscillator system which is covariant under some quantum groups. The first work about the orthosymplectic invariance for the undeformed supersymmetric phase variables was accomplished by

Casabuoni [12]. The system of n q-bosons covariant under $U_q(n)$ was first presented by Pusz and Woronowicz [13] and the system of n q-bosons and m q-fermions by Chaichian *et al* [14].

In this paper, we introduce the q-deformed boson–fermion covariance which has $sl_q(n|n)$ covariance and construct the q-deformed $N = 2$ SUSY algebra with $sl_q(n|n)$ covariance.

2. q-SUSY with $sl_q(1|1)$ covariance

Before dealing with the general $sl_q(n|n)$ case, let us consider the $n = 1$ case in this section, which implies the system of one q-boson and one q-fermion among which there exists $sl_q(1|1)$ covariance.

If we impose the $sl_q(1|1)$ covariance for a single-mode q-boson and q-fermion, the algebra is given by

$$\begin{aligned} &af = \sqrt{q} fa \qquad f^\dagger a^\dagger = \sqrt{q} a^\dagger f^\dagger \\ &af^\dagger = \sqrt{q} f^\dagger a \qquad fa^\dagger = \sqrt{q} a^\dagger f \qquad f^2 = (f^\dagger)^2 = 0 \\ &aa^\dagger = 1 + qa^\dagger a \\ &ff^\dagger = 1 - f^\dagger f + (q-1)a^\dagger a \end{aligned} \tag{1}$$

where we have assumed that q is real and positive. Here, $f^\dagger$ (or f) is a creation (or annihilation) operator for a q-fermion and $a^\dagger$ (or a) is a creation (or annihilation) operator for a q-boson. The first, second and third relations of equation (1) show that the step operators for the q-fermion and q-boson are not mutually independent but q-commutative. The fourth relation shows that these systems still obeys the Pauli exclusion principle. The fifth relation is the well known single-mode q-boson algebra. The sixth relation is more or less unknown to us, but the last term is indispensable for $sl_q(1|1)$ invariance of the algebra (1).

It can easily be shown that the algebra given in equation (1) remains invariant under the $sl_q(1|1)$ transformation

$$\begin{aligned} &\begin{pmatrix} a' \\ f' \end{pmatrix} = \begin{pmatrix} A & \beta \\ -(A^*)^{-1}\beta^*(A^*)^{-1} & (A^*)^{-1} \end{pmatrix} \begin{pmatrix} a \\ f \end{pmatrix} \\ &\left((a^\dagger)' \quad (f^\dagger)' \right) = \left(a^\dagger \quad f^\dagger \right) \begin{pmatrix} A^* & -A^{-1}\beta A^{-1} \\ \beta^* & A^{-1} \end{pmatrix} \end{aligned} \tag{2}$$

where the 2×2 matrix is a quantum super-matrix for $sl_q(1|1)$ algebra and its entries obey the following commutation relation:

$$\begin{aligned} &A\beta = \sqrt{q}\beta A \qquad A^*\beta \frac{1}{\sqrt{q}} \beta A^* \\ &\beta\beta^* + q\beta^*\beta = 0 \qquad \beta^2 = (\beta^*)^2 = 0 \\ &AA^* - A * A = (q-1)\beta^*\beta \\ &AA^* + \beta\beta^* = 1. \end{aligned} \tag{3}$$

Here $*$ implies complex conjugation and it is assumed that (A, A^*) commute with $(a, a^\dagger)$ and $(f, f^\dagger)$, and (β, β^*) commute with $(a, a^\dagger)$ and anticommute with $(f, f^\dagger)$.

The commutation relation between a and f is given by the following R-matrix for $sl_q(1|1)$:

$$\begin{pmatrix} a \\ f \end{pmatrix} \otimes \begin{pmatrix} a \\ f \end{pmatrix} = \hat{R} \begin{pmatrix} a \\ f \end{pmatrix} \otimes \begin{pmatrix} a \\ f \end{pmatrix}$$

where

$$\hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-1} & q^{-1/2} & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4}$$

is the R-matrix for $sl_q(1|1)$. It is well known that the R-matrix for $sl_q(1|1)$ satisfies the Yang–Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{5}$$

which implies the associativity and $sl_q(1|1)$-covariance of the algebra (1).

Then, there exist two types of supercharges as follows:

$$Q = \frac{1}{\sqrt{q}} a^\dagger f \qquad Q^\dagger = \frac{1}{\sqrt{q}} f^\dagger a \tag{6}$$

where the nilpotency of f and $f^\dagger$ gives the nilpotency of q-supercharges

$$Q^2 = (Q^\dagger)^2 = 0. \tag{7}$$

Now we construct the q-SUSY quantum mechanics for this system. Before doing this, we introduce number operators for this system. Let N and M be number operators for bosons and q-fermions, respectively. We then have the following relations:

$$\begin{aligned} &[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \\ &[M, f^\dagger] = f^\dagger \qquad [M, f] = -f \end{aligned} \tag{8}$$

where the relation between N and a ($a^\dagger$) is given by

$$[N] = a^\dagger a \tag{9}$$

or

$$N = \sum_{k=1}^{\infty} \frac{(1-q)^k}{(1-q^k)} (a^\dagger)^k a^k. \tag{10}$$

Here the q-number $[x]$ is defined as

$$[x] = \frac{q^x - 1}{q - 1}.$$

Similarly, the relation between the q-fermion number operator and q-fermion step operators are given by

$$f^\dagger f = q^N M \tag{11}$$

or

$$M = f^\dagger f q^{-N}. \tag{12}$$

If we use the nilpotency of the q-fermion step operators, we have $[M] = M$, so we have the following relation:

$$a^\dagger a + f^\dagger f = [N + M]. \tag{13}$$

If we introduce the Fock basis $|nm\rangle$ for N and M as follows:

$$\begin{aligned} &N|n, m\rangle = n|n, m\rangle \\ &M|n, m\rangle = m|n, m\rangle. \end{aligned} \qquad (n = 0, 1, 2, \ldots, m = 0, 1) \tag{14}$$

The matrix representation of a, $a^\dagger$, $f^\dagger$ and f are given by

$$
\begin{aligned}
&a|n,m\rangle = \sqrt{[n]}|n-1,m\rangle \\
&a^\dagger|n,m\rangle = \sqrt{[n+1]}|n+1,m\rangle \\
&f^\dagger|n,1\rangle = 0 \qquad f^\dagger|n,0\rangle = q^n|n,1\rangle \\
&f|n,0\rangle = 0 \qquad f|n,1\rangle = q^n|n,0\rangle.
\end{aligned} \tag{15}
$$

Equation (5) implies that there exists a ground state $|0,0\rangle$ killed by both a and f and that any excited states can be obtained by applying the raising operators successively to the ground states.

Using the Fock representation, the q-SUSY algebra becomes

$$
\begin{aligned}
&\{Q, Q^\dagger\}_q = H \\
&[Q, H]_q = [H, Q^\dagger]_q = 0 \\
&Q^2 = (Q^\dagger)^2 = 0
\end{aligned} \tag{16}
$$

where the Hamiltonian H is given by

$$H = q^{-1}a^\dagger a + q^{-1}(q-1)(a^\dagger a)^2 + q^{-2} f^\dagger f. \tag{17}$$

Here q-brackets are defined as follows:

$$
\begin{aligned}
&[X, Y]_q = qXY - q^{-1}YX \\
&\{X, Y\}_q = qXY + q^{-1}YX.
\end{aligned}
$$

It is worth noting that the q-analogue of $N = 2$ SUSY algebra given in equation (16) is the same as that introduced by Spiridonov [7]. Some of its physical application is given in [7].

This Hamiltonian has two quadratic terms corresponding to the q-bosons and q-fermions which implies the kinetic terms for this system. The second term of the Hamiltonian (17) is more or less unknown to us, but it comes about because of q-deformed SUSY algebra.

Acting the supercharges on the Fock basis of number operators, we have

$$
\begin{aligned}
&Q|n,0\rangle = 0 \qquad Q|n,1\rangle = q^{n/2}\sqrt{[n+1]}|n+1,\downarrow\rangle \\
&Q^\dagger|n,1\rangle = 0 \qquad Q^\dagger|n,0\rangle = q^{n/2}\sqrt{[n]}|n-1,\uparrow\rangle.
\end{aligned} \tag{18}
$$

The energy eigenvalues for this Hamiltonian become

$$
\begin{aligned}
&H|n,1\rangle = q^n[n+1]|n,1\rangle \\
&H|n,0\rangle = q^{n+1}[n]|n,0\rangle.
\end{aligned} \tag{19}
$$

Thus, the relation between the Hamiltonian and number operators are given by

$$H = q^N(q[N] + [M]). \tag{20}$$

3. q-SUSY with $sl_q(n|n)$ covariance

Let us consider the general n case in this section, which implies the system of n q-bosons and n q-fermions among which there exists $sl_q(n|n)$ covariance.

If we impose the $sl_q(n|n)$ covariance for n q-bosons and n q-fermions, the algebra is given by

$$
\begin{aligned}
&a_i a_j = q^{1/2} a_j a_i && (i < j)\\
&a_i a_j^\dagger = q^{-1/2} a_j^\dagger a_i && (i < j)\\
&a_i a_j^\dagger = q^{1/2} a_j^\dagger a_i && (i \neq j)\\
&a_i a_i^\dagger = 1 + q a_i^\dagger a_i + (q-1)\sum_{k=1}^{i-1} a_k^\dagger a_k &&\\
&a_i f_j = \sqrt{q} f_j a_i && f_j^\dagger a_i^\dagger = \sqrt{q} a_i^\dagger f_j^\dagger\\
&a_i f_j^\dagger = \sqrt{q} f_j^\dagger a_i && f_j a_i^\dagger = \sqrt{q} a_i^\dagger f_j\\
&f_i f_j = -\sqrt{q} f_j f_i && (i < j)\\
&f_i^\dagger f_j^\dagger = -q^{-1/2} f_j^\dagger f_i^\dagger && (i < j)\\
&f_i f_j^\dagger = -\sqrt{q} f_j^\dagger f_i && (i \neq j)\\
&f_i f_i^\dagger + f_i^\dagger f_i = 1 + (q-1)\left(\sum_{k=1}^{n} a_i^\dagger a_i + \sum_{k=1}^{i-1} f_k^\dagger f_k\right) &&\\
&f_k^2 = (f_k^\dagger)^2 = 0. &&
\end{aligned}
\tag{21}
$$

It can be easily checked that the algebra (21) is covariant under the $sl_q(n|n)$ transformation of the step operators of q-bosons and q-fermions.

Then, there exist $2n$ types of supercharges as follows:

$$
Q_i = \frac{1}{\sqrt{q}} a_i^\dagger f_i \qquad Q_i^\dagger = \frac{1}{\sqrt{q}} f_i^\dagger a_i \tag{22}
$$

where the nilpotency of f_i and $f_i^\dagger$ gives the nilpotency of supercharges

$$
Q_i^2 = (Q_i^\dagger)^2 = 0. \tag{23}
$$

If we introduce the Fock basis $|\boldsymbol{n}, \boldsymbol{m}\rangle$ for N_i and M_i as follows:

$$
\begin{aligned}
N_i|\boldsymbol{n}, \boldsymbol{m}\rangle &= n_i|\boldsymbol{n}, \boldsymbol{m}\rangle\\
M_i|\boldsymbol{n}, \boldsymbol{m}\rangle &= m_i|\boldsymbol{n}, \boldsymbol{m}\rangle
\end{aligned}
\qquad (n_i = 0, 1, 2, \ldots, m_i = 0, 1) \tag{24}
$$

the matrix representation of a_i, $a_i^\dagger$, f_i and $f_i^\dagger$ is given by

$$
\begin{aligned}
a_i|\boldsymbol{n}, \boldsymbol{m}\rangle &= \sqrt{q^{\sum_{k=1}^{i-1} n_k}[n_i]}\,|\boldsymbol{n} - \boldsymbol{e}_i, \boldsymbol{m}\rangle\\
a_i^\dagger|\boldsymbol{n}, \boldsymbol{m}\rangle &= \sqrt{q^{\sum_{k=1}^{i-1} n_k}[n_i + 1]}\,|\boldsymbol{n} + \boldsymbol{e}_i, \boldsymbol{m}\rangle\\
f_i|\boldsymbol{n}, \boldsymbol{m}\rangle &= \begin{cases} 0 & (m_i = 0)\\ q^{\frac{1}{2}\sum_{k=1}^{n} n_k}(-q^{1/2})^{\sum_{k=1}^{i-1} m_k} m_i|\boldsymbol{n}, \boldsymbol{m} - \boldsymbol{p}_i\rangle & (m_i = 1)\end{cases}\\
f_i^\dagger|\boldsymbol{n}, \boldsymbol{m}\rangle &= \begin{cases} q^{\frac{1}{2}\sum_{k=1}^{n} n_k}(-q^{1/2})^{\sum_{k=1}^{i-1} m_k}(1 - m_i)|\boldsymbol{n}, \boldsymbol{m} + \boldsymbol{p}_i\rangle & (m_i = 0)\\ 0 & (m_i = 1)\end{cases}
\end{aligned}
\tag{25}
$$

where we have used the following abbreviations:

$$
\begin{aligned}
|\boldsymbol{n}, \boldsymbol{m}\rangle &= |n_1, \ldots, n_n, m_1, \ldots, m_n\rangle\\
|\boldsymbol{n} \pm \boldsymbol{e}_i, \boldsymbol{m}\rangle &= |n_1, \ldots, n_{i-1}, n_i \pm 1, n_{i+1}, \ldots, n_n, m_1, \ldots, m_n\rangle\\
|\boldsymbol{n}, \boldsymbol{m} \pm \boldsymbol{p}_i\rangle &= |n_1, \ldots, n_n, m_1, \ldots, m_{i-1}, m_i \pm 1, m_{i+1}, \ldots, m_n\rangle.
\end{aligned}
$$

Now we construct the $N = 2$ q-SUSY algebra for this system. The q-SUSY algebra with $sl_q(n|n)$ covariance is given by

$$\begin{aligned}
&\{Q_i, Q_j\} = 0 \\
&\{Q_i, Q_j\}_q = H_i \\
&\{Q_i, Q_j^\dagger\}_{\sqrt{q}} = 0 \qquad (i < j) \\
&[Q_i, H_i]_q = 0 \\
&[Q_j, H_i]_{\sqrt{q}} \qquad (i \neq j)
\end{aligned} \tag{26}$$

where the sub-Hamiltonians H_i are given by

$$\begin{aligned}
H_i = {} & q^{-1} a_i^\dagger a_i + q^{-2} f_i^\dagger f_i + q(q-1) a_i^\dagger a_i \left(\sum_{k=1}^{n} a_k^\dagger a_k + q^{-2} \sum_{k=1}^{i-1} f_k^\dagger f_k \right) \\
& + q^{-2}(q-1) f_i^\dagger f_i \sum_{k=1}^{i-1} a_k^\dagger a_k .
\end{aligned} \tag{27}$$

These sub-Hamiltonians have two quadratic terms corresponding to two q-bosons and two q-fermions which imply the kinetic terms for this system. The remaining terms of the sub-Hamiltonians (27) are more or less unknown to us, but they come about because of the q-deformed SUSY algebra.

4. Conclusion

In this paper I have studied the q-deformed SUSY for q-bosons and q-fermions among which there exists $sl_q(n|n)$ covariance. In particular, I have discussed the $n = 1$ case more explicitly. In contrast to the earlier deformation of $N = 2$ SUSY, the q-bosons and q-fermions are not independent but q-commutative in this model, which results from the fact that this system consisting of q-bosons and q-fermions possesses $sl_q(n|n)$ covariance. Using the q-deformed $N = 2$ SUSY with $sl_q(n|n)$ covariance, I have discussed the Hamiltonian for these (strange) particles.

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References

[1] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[2] Drinfeld V 1986 *Proc. Int. Congress of Mathematicians (Berkeley, CA)* p 78
[3] Biedenharn L 1989 *J. Phys. A: Math. Gen.* **22** L873
[4] Macfarlane A 1989 *J. Phys. A: Math. Gen.* **22** 4581
[5] Parthasarathy R and Viswanathan K 1991 *J. Phys. A: Math. Gen.* **24** 613
[6] Witten E 1981 *Nucl. Phys.* B **185** 513
[7] Spiridonov V 1992 *Mod. Phys. Lett.* A **7** 1241
[8] Chung W 1995 *Prog. Theor. Phys.* **94** 649
[9] Schmidke W, Vokos S and Zumino B 1990 *Z. Phys.* C **48** 249
[10] Soni S 1991 *J. Phys. A: Math. Gen.* **24** L459
[11] Burdik C and Tomasek R 1992 *Lett. Math. Phys.* **26** 97
[12] Casalbuoni R 1976 *Nuovo Cimento* A **33** 124
[13] Pusz W and Woronowicz S 1989 *Rep. Math. Phys.* **27** 231
[14] Chaichian M, Kulish P and Lukierski J 1991 *Phys. Lett.* B **262** 43